Thermal Convection in Non-Newtonian Fluids

An alternate look at the Rayleigh-Bénard problem

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Abstract

We study the onset of Rayleigh-Bénard instability in a thin layer of a viscous non-Newtonian (shear-thinning) fluid. Through a weakly nonlinear analysis we derive the cubic-quintic Ginzburg-Landau equation describing the temporal evolution of small amplitude convection. Analysis of the stability of spatially periodic solutions enables us to determine regions of Eckhaus and oscillatory instability in parameter space.

Key words: Rayleigh-Bénard experiment, non-Newtonian fluids, bifurcation theory, weakly non-linear analysis, multiple scales theory, stability theory.

Introduction

Rayleigh-Bénard convection occurs when a horizontal layer of fluid, bounded from above and below by horizontal plates, is heated from below, see Figure 1. Convective motion arises when the temperature difference, ΔT, between the two plates is increased past a critical value, which is dependent on the Rayleigh number, R. Rayleigh-Bénard convection has been the subject of intense experimental and theoretical work over more than a century, due to the ubiquity of thermally-driven flows in industrial hydrodynamics, atmospheres, and ocean dynamics. In this poster we consider weakly non-linear thermal convection in a non-Newtonian fluid. We derive and then study the properties of the amplitude equation describing the small-amplitude behaviour of convection rolls.

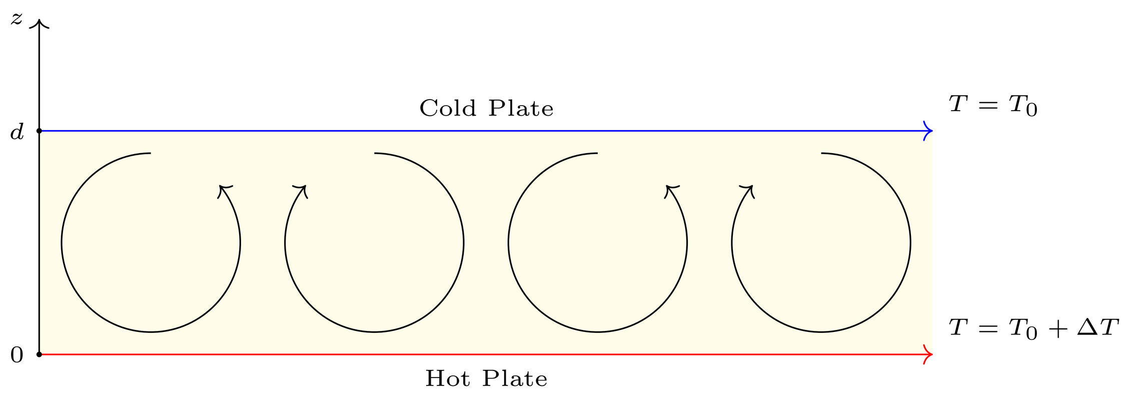

Figure 1: 2D representation of the theoretical experiement set up. The fluid is confined between two plates separated by a distance d, spanning infinitely in the plane with vertical (normal) axis z.

Methods: Formulation

The conservation laws, including the heat diffusion equation, for a non-Newtonian shear-thinning viscoplastic fluid are:

- Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{1}$$

- Conservation of Momentum:

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \cdot \boldsymbol{u} \right) = -\nabla p + \rho \boldsymbol{g} + \rho \nu_0 \nabla \cdot \boldsymbol{\tau} \tag{2}$$

- Heat Diffusion Equation:

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa_T \nabla^2 T \tag{3}$$

where $\boldsymbol{u}$ is the velocity of the fluid, T is the temperature of the fluid, $\boldsymbol{g} = (0, 0, -g)$ is the acceleration due to gravity, ν_0, ν_∞ is the viscosity of the fluid at zero and infinite shear rate respectively, $\boldsymbol{\tau}$ is the characteristic shear stress and κ_T is the thermal diffusivity.

We set $\boldsymbol{\tau} = \mu(\Gamma)\dot{\boldsymbol{\gamma}}$, given $\Gamma = -\mathrm{tr}\left(\frac{1}{2}\boldsymbol{\gamma}^T\boldsymbol{\gamma}\right)$ and $\mu(\Gamma) = \frac{\nu(\Gamma)}{\nu_0}$ where $\nu(\Gamma)$ satisfies the Carreau model, $\dot{\boldsymbol{\gamma}} := \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T$ is the shear rate. We find that we may assume $\mu(\Gamma) := 1 + \nu_d \Gamma$, where $\nu_d < 0$ is the degree of shear thinning [1].

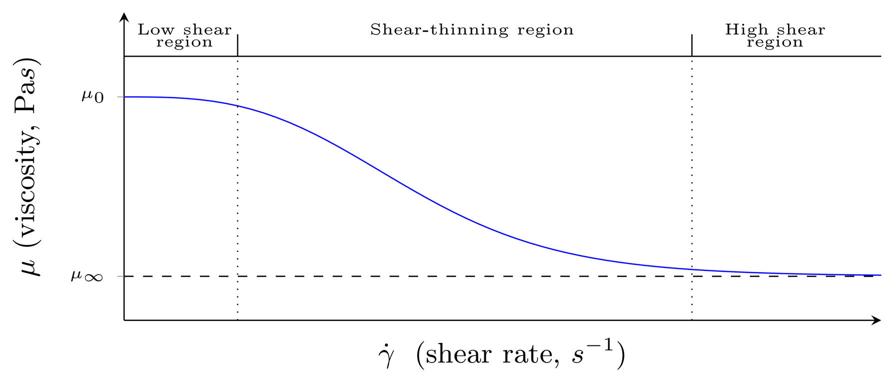

Figure 2: The constitutive law for the Carreau model.

The flow is driven by density variations induced by the temperature gradient. For simplicity, we assume that the density varies linearly in T as:

$$\rho(T) = \rho_0(1 - \alpha_T(T - T_0)). \tag{4}$$

We have further adopted the Boussinesq approximation, forcing the density to vary only on buoyant terms and otherwise stay constant; reducing (1) to $\nabla \cdot \boldsymbol{u} = 0$.

By considering small perturbations to the static solutions of p and T, and defining the velocity as:

$$\boldsymbol{u} = (-\psi_z, 0, \psi_x)$$

we obtain a system of governing equations:

$$\theta_t + J_{xz}(\psi, \theta) = \psi_x + \nabla^2\theta \tag{5}$$

$$\frac{1}{\sigma R_c}\left((\nabla^2\psi)_t + J_{xz}(\psi, \nabla^2\psi)\right) = r\theta_x + \frac{1}{R_c}\nabla^4\psi + \frac{1}{R_c}N \tag{6}$$

where θ is the thermal perturbation, $r = \frac{R}{R_c}$, $\sigma = \frac{\nu}{\kappa_T}$ is the Prandtl number, $R = \frac{\alpha_T \Delta T g d^3}{\nu \kappa_T}$ is the Rayleigh number and $N = N(\Gamma)$ is the contribution due to non-Newtonian effects.

Boundary Conditions:

We make the following assumptions:

(i) Fixed temperature at the plates;
(ii) Periodic boundary in x with no lateral heat flux at $x = 0, L$;
(iii) Impermeable boundaries;
(iv) Stress-free walls;
(v) Resting state is not pre-stressed.

Weakly Non-Linear Analysis

To study instabilities close the critical Rayleigh number R_c, we rescale the time and spacial scales whilst perturbing selected variables according to: $\frac{\partial}{\partial t} \to \epsilon^4 \frac{\partial}{\partial t_4}$, $X = \epsilon^2 x$, $r = 1 + \epsilon^4 r_4$ and $\nu_d = \nu_d^c(1 + \epsilon^2\nu_2)$. Now due to the non-linearly of the system, we asymptotically expand up to firth order the ψ, θ terms according to:

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots \quad \text{and} \quad \theta = \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots$$

to obtain a system of tractable equations, with

$$\psi_1 = \frac{2\sqrt{2}\beta_0}{\alpha}\sin(\pi z)(Ae^{i\alpha x} + \bar{A}e^{-i\alpha x}).$$

Solving the equations obtained at each order of ϵ, we obtain at fifth order an envelope equation describing the temporal evolution of the convection amplitude, which has the form:

$$\begin{aligned} A_T = \mu A + A_{XX} + i\left(a_1|A|^2A_X + a_2A^2\bar{A}_X\right) \\ + b|A|^2A - |A|^4A \end{aligned} \tag{7}$$

where:

$$\mu = \frac{4r_4}{k_1^{1/5}}, b = \frac{1202\nu_2\pi^4}{3k_1^{3/5}},$$

$$a_1 = \frac{\sqrt{3}}{4\sqrt{k_1}}\left(\frac{7}{4\sigma} + \frac{10}{3\sigma^2} - k_3\right), a_2 = \frac{\sqrt{3}}{4\sqrt{k_1}}\left(\frac{7}{4\sigma} + \frac{10}{3\sigma^2} - k_2\right),$$

and k_i are determined positive constants. Having obtained the amplitude equation, [2] states that in a normal form approximation for the Hamiltonian-Hopf bifurcation of the extended SH equation near the codimension-two point $(\mu, q_2) = (0, 0)$, the dynamics are strongly dependent on a normal form coefficient q_4, which has the following relation to the amplitude equation obtained above:

$$\begin{aligned} A_{XX} = -q_1\mu A + q_2A|A|^2 + i\left(p_2 + \frac{1}{2}q_3\right)A^2\bar{A}_X + \\ i\left(3p_2 - \frac{1}{2}q_3\right)A_X|A|^2 + (q_4 - q_3p_2 + p^2)A|A|^4 \end{aligned} \tag{8}$$

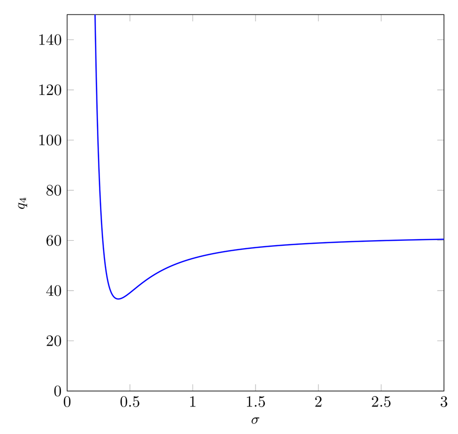

Figure 3: (σ, q_4) plot which shows that $q_4 > 0$ for all $\sigma > 0$.

Equating coefficients leads to the conclusion that $q_4 > 0$ (see Figure 3), which from Figure 4 means that the dynamics of our system follow the regime outlined by a). We note that here $f(y) = 4q_1\mu y^2 - 2q_2y^3 - \frac{4}{3}q_4y^4$.

Stability Analysis

Stationary solutions to the CQ-GLE yield two conserved quantities E and L. Solutions of the form: $A(X) = R(X)e^{i\phi(X)}$ satisfy $R_X^2 + U = E$, where $U = U(R; \mu, L)$, reducing the problem to considering a particle in a well of potential U and total energy E.

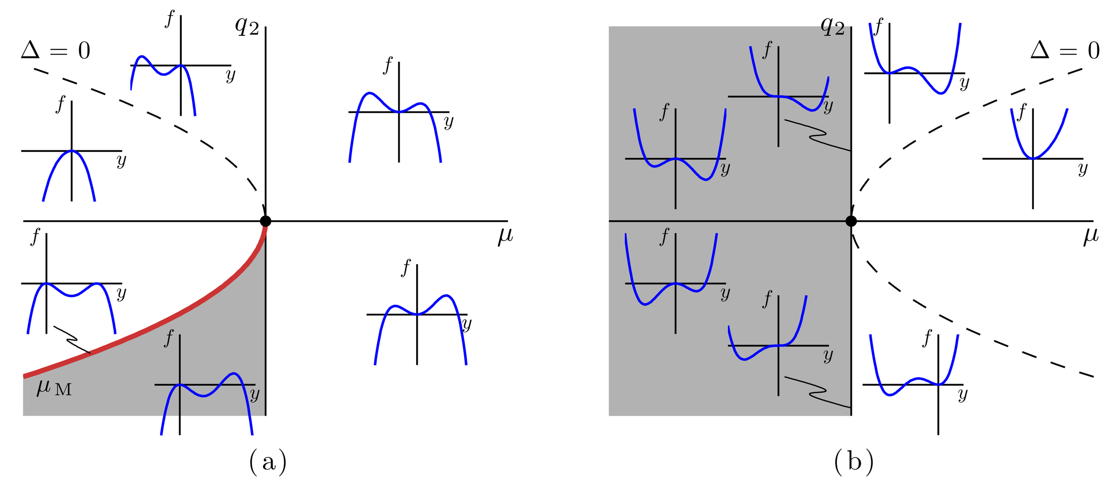

Figure 4: Extract from [2]: *Schematic summary of the regimes of normal form dynamics in the (μ, q_2) plane, in the cases a) $q_4 > 0$ and b) $q_4 < 0$. ... Orbits homoclinic to the origin exist in the shaded regions. The dashed curves indicate $\Delta = 0$ where the quantity $\Delta \equiv q_2^2/4 + 4q_1q_4\mu/3 = 0$ is the discriminant of $f(y)/y^2$. The solid line in (a) labelled μ_M denotes the Maxwell point.*

The paper was to study the stability properties of equilibrium points. The loss of stability of an equilibrium point corresponds to the appearance of new steady states with an X-dependent amplitude R and phase ϕ. Such bifurcations occur at amplitudes R_0 defined by $U_R = 0$ satisfying $U_{RR} > 0$. Considering $R = R_0(k)$ with $\phi_X = k$, these satisfy the polynomial equation $\mu - k^2 + [b + k(a_2 - a_1)]R_0^2 - R_0^4 = 0$. If we let $b' := b + k(a_2 - a_1)$, then $b' \leq 0$ corresponds a supercritical bifurcation, where only the R_0^+ branch exists and $b' > 0$ corresponds to a subcritical bifurcation where R_0^+ is stable and R_0^- is unstable.

Results

We then studied the stability properties of constant amplitude steady solutions of the form $A = R_0e^{i(kx+\phi_0)}$ on primary branches k w.r.t. long-wave perturbations with wavenumber q, where $|q| \ll |k|$. We perturbed to solutions as follows $A = R_0e^{i(kX+\phi_0)}(1 + a)$, for $a \equiv a(X, T) := \beta_1(T)e^{iqX} + \bar{\beta}_2(T)e^{-iqX}$, yielding the eigenvalues: $\lambda_\pm = -(g + q^2) \pm \sqrt{(g + q^2)^2 - q^2(f + q^2)}$, for known functions f, g [3]. From this, we deduced a regions of Eckhaus and oscillatory instabilities in (k, μ) parameter space for a variety of parameter vales, some of which can be seen below:

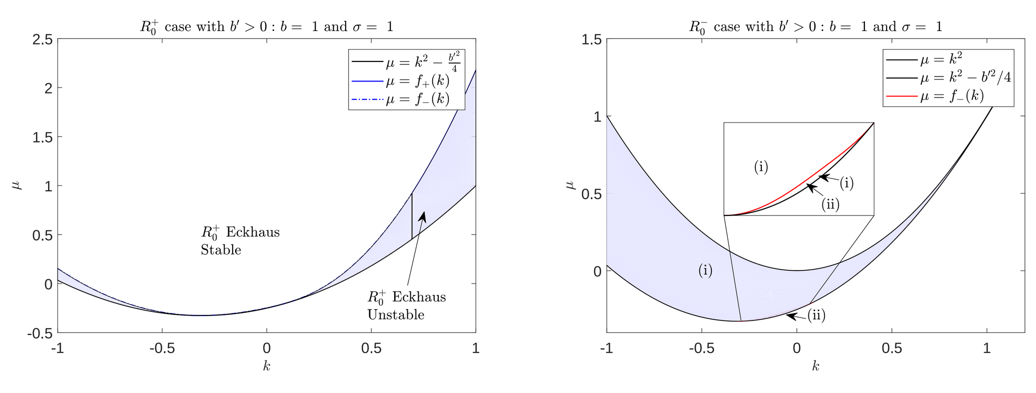

Figure 5: Stability plots in (k, μ) parameter space, where shaded regions indicate regions of instability. We note that type (i) instability corresponds to instability formed when $f < 0$ and type (ii) instability corresponds to instability formed when $g < \min\{0, f\}$. It can be seen that in the right hand plot a small region of type (ii) instabilities exists. Here $f_\pm(\mu)$ denotes the positive and negative roots of the quadratic in μ obtained when evaluating the inequalities.

We note that oscillatory instabilities cannot occur on the R_0^+ branch in any case, instead, only Eckhaus stabilities are present.

References

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