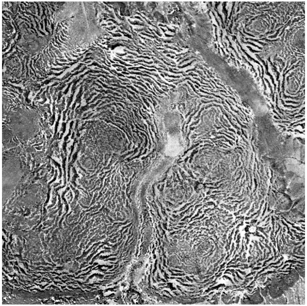
Pattern Formation in Fluids

Anvarbek Atayev

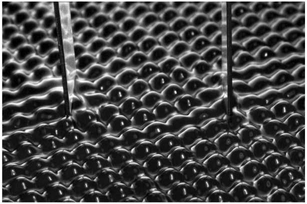
University of Bath

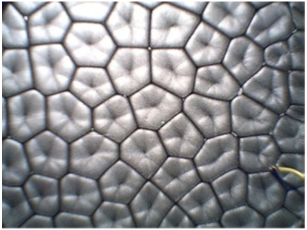
November 14, 2017

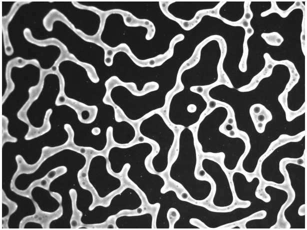
Universality of Patterns



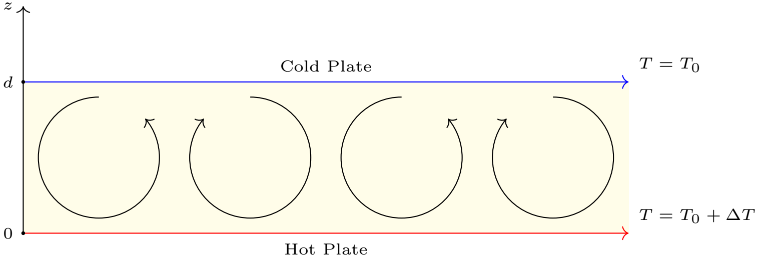
Patterns on Surfaces of Fluids



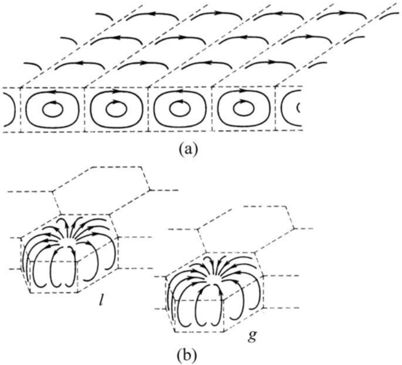


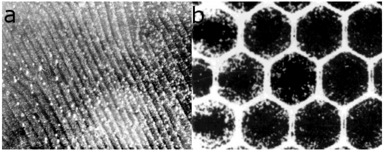


Rayleigh Bénard Experiment



Rayleigh Bénard Experiment II





Cooking Hexagons

Ingredients: Oil, Black Pepper. Equipment: Flat Bottomed Pan, Stove.

1. Pour a very thin layer of oil into the pan $\approx$ 2mm.
2. Add a dash of fine black pepper, enough to finely cover your layer of oil.
3. Put the pan on a flat, somewhat uniform, heat source and heat gently.
4. Look at the oil from the side and you should observe that hexagons have formed.

Voilà!

Example

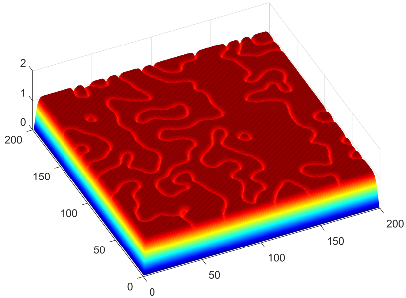
Let's look at the following dynamical system:

$$u_t = D\nabla^2 u + \gamma q(u)u, \tag{1}$$

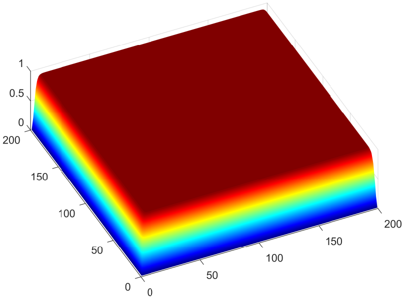
under random initial conditions, where

$$q(u) = u(1-u). \tag{2}$$

Example II



(a) $D = 0.2, \gamma = 0.5.$

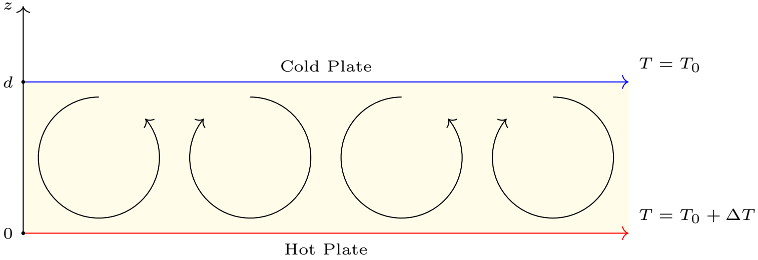


(b) $D = 0.2, \gamma = 0.2.$

Non-Newtonian Fluids

- Starch + Water;
- Toothpaste;
- Porridge;
- Earth's mantle;
- etc..

My Project: Set-Up + Model



My Project: Set-Up + Model II

- Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{3}$$

- Conservation of Momentum:

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + \rho g + \rho \nu_0 \nabla \cdot \tau \tag{4}$$

- Heat Equation:

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa_T \nabla^2 T \tag{5}$$

My Project: Set-Up + Model III

Model:

$$\tau = \mu(\Gamma)\dot{\gamma}, \quad \Gamma = \mathrm{tr}\left(\frac{1}{2}\dot{\gamma}^T\dot{\gamma}\right) \quad \mu(\Gamma) = \frac{\nu(\Gamma)}{\nu_0} \tag{6}$$

$$\frac{\nu(\Gamma) - \nu_\infty}{\nu_0 - \nu_\infty} = (1 + \lambda^2\Gamma)^{\frac{n_c - 1}{2}} \tag{7}$$

We approximate μ as:

$$\mu(\Gamma) = \frac{\nu(\Gamma)}{\nu_0} \approx \frac{\nu(\Gamma) - \nu_\infty}{\nu_0 - \nu_\infty} = 1 + \nu_d\Gamma, \tag{8}$$

where $\nu_d = \left(\frac{n_c - 1}{2}\lambda^2\right)$. [Insert Carreau model here.]

My Project: Set-Up + Model IV

Now let T_s and p_s be the static solutions. Then we consider perturbations to these static solutions as follows:

$$T = T_s + \Delta T\theta$$
$$p = p_s + Q\tilde{p}$$

As we have incompressibility, we may write u as:

$$u = (u, 0, w) = (-\psi_z, 0, \psi_x). \tag{9}$$

Let $\Omega := \nabla \times u = (0, \omega, 0)$, then $\omega = \nabla^2\psi$.

My Project: Set-Up + Model V

Rewriting the equations is terms of the heat perturbation and wavefunction, we obtain our set of governing equations:

$$\theta_t + J_{x,z}(\psi, \theta) = \psi_x + \nabla^2 \theta \tag{10}$$

$$\frac{1}{\sigma R_c}\left(\omega_t + J_{x,z}(\psi, \omega)\right) = r\theta_x + \frac{1}{R_c}\nabla^2 \omega + \frac{1}{N_c} N, \tag{11}$$

where $N := (\partial_{xx} - \partial_{zz})(\nu_d \Gamma(\psi_{xx} - \psi_{zz})) + 4\partial_{xz}(\nu_d \Gamma \psi_{xz})$

We perturb certain variables and consider long time and long spatial scales by applying the following:

$$\frac{\partial}{\partial t} \to \epsilon^4 \frac{\partial}{\partial t_2}, \quad X = \epsilon^2 x, \quad r = 1 + \epsilon^4 r_2, \quad \nu_d = \nu_d^c (1 + \epsilon^2 \nu_2). \tag{12}$$

Now we study the behaviour of our system when the fluid starts to change from a resting state, therefore:

$$\psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots \tag{13}$$

$$\theta = \epsilon \theta_1 + \epsilon^2 \theta_2 + \cdots . \tag{14}$$

Now letting: $\psi_1 = \frac{2\sqrt{2}\beta_0}{\alpha} \sin(\pi z)(A e^{i\alpha x} + \bar{A} e^{-i\alpha x})$, we aim to solve the above system up to fifth order.

At fifth order, we obtain the following amplitude equation:

$$A_T = \mu A + A_{XX} + i\left(a_1|A|^2A_X + a_2A^2\bar{A}_X\right) + b|A|^2A - |A|^4A. \qquad (15)$$

My Project: Stationary Solutions

We see to find stationary solutions to this equation. It can be shown that there exist two conserved quantities:

$$E \equiv (\mu + 2a_2 L)|A|^2 + |A_X|^2 + \frac{b}{2}|A|^4 - \left(\frac{1}{3} + \frac{a_2(a_1 + a_2)}{6}\right)|A|^6$$

$$L \equiv \frac{i}{2}(A\bar{A}_X - \bar{A}A_X) + \frac{a_1 + a_2}{4}|A|^4$$

If we consider rolling wave solutions, $A = R(X)e^{i\phi(X)}$, then the system is reduced to a particle in a well problem:

$$E = R_X^2 + U$$

$$U(R;\mu, L) \equiv \frac{L^2}{R^2} + \left(\mu + \frac{3a_2 - a_1}{2}L\right)R^2 + \frac{b}{2}R^4 + \beta R^6$$

My Project: Stationary Solutions

Now consider stationary solutions to this equation of the form: $A = R_0 e^{i\phi(X)}$. These are roll solutions. Our aim becomes to determine the stability of these solutions. We can do this by considering small perturbations about this solution of the form:

$$A = R(X)e^{i\phi(X)}(1 + a), \tag{16}$$

where $a(X, T) = \beta_1(T)e^{iqX} + \bar{\beta}_2(T)e^{-iqX}$.

My Project: Stability of Solutions

We can solve the previous system by linearising to something of the from:

$$\frac{\mathrm{d}}{\mathrm{d}T}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix} - & - \\ - & - \end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \tag{17}$$

which has eigenvalues:

$$\lambda_{\pm} = -(g+q^2) \pm \sqrt{(g+q^2)^2 - q^2(f+q^2)},$$

where: $f(\mu,k) = (2 + a_2^2 - a_1^2)(\mu - k^2 + b' R_0^2) + 2\mu - 6k^2 - 4ka_1 R_0^2$, $g(\mu,k) = 2(\mu - k^2) + b' R_0^2$, and $b' = b + k(a_2 - a_1)$.

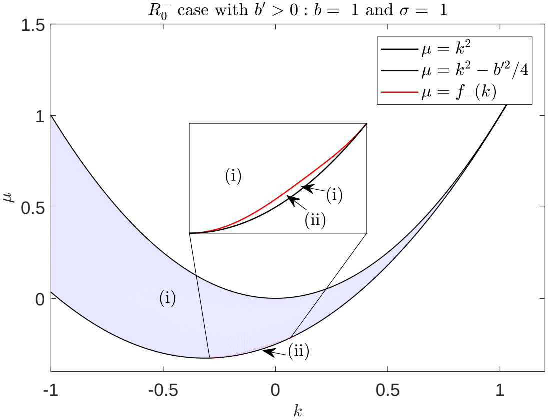
My Project: Stability Curves

Now we observe two types of instabilities:

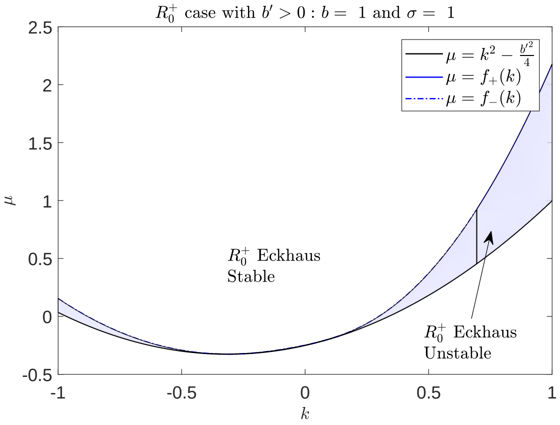
(i) $f < 0$;

(ii) $g < \min\{0, f\}$.

My Project: Stability Curves



My Project: Stability Curves



Future

We aim to study the fully non-linear case by numerically solving the set of governing PDE's.

References

1. M.I. Rabinovich, A.B. Ezersky, P.D. Weidman; The Dynamics of Patterns
2. R. Hoyle; Pattern Formation
3. A. Atayev, J. Dawes Pattern Formation in Thermally Convecting Non-Newtonian Fluids